Unsteady heat transfer for flow over a flat plate

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The boundary-layer flow over a semi-infinite flat plate is investigated. For time $t < 0$ there is the usual steady velocity boundary layer and, neglecting viscous dissipation, no thermal boundary layer. At $t = 0$ a temperature boundary layer is initiated without altering the velocity, and the subsequent temperature distribution is studied for large and small t.

1. Introduction

In a recent paper Cess (1962) considers the heat transfer from a semi-infinite flat plate immersed in an incompressible fluid which, at infinity, has constant velocity U_∞ parallel to the plate. For time $t < 0$ the plate and the fluid have the same temperature T_∞. At $t = 0$ the plate temperature is changed so that for $t > 0$ the plate is maintained at a constant temperature $T_w \neq T_\infty$. The applied heating is assumed to be sufficiently small for the consequent changes in density of the fluid to have no effect upon the steady velocity distribution in the boundary layer already established on the plate; the Mach number of the flow is also assumed to be so small that the effects of viscous dissipation may be neglected. Cess considers fluids with Prandtl numbers which are small so that when the steady state is reached the thermal boundary layer is very much thicker than the velocity boundary layer, and throughout his analysis he replaces the components of velocity occurring in the temperature equation by their values in the outer region of the velocity boundary layer. In this way he develops a series solution, for small τ, in powers of $\tau^{\frac{1}{2}}$ where $\tau = U_\infty t/x$, x being the distance along the plate measured from its leading edge. This procedure he justifies as follows: 'during the initial stages of the heat-transfer process the influence of convection will be small, such that any error in the formulation of the velocities should not appreciably affect the overall results'.

However it is clear that, regardless of the value of the Prandtl number, when τ is sufficiently small the thermal boundary layer is very much thinner than the velocity boundary layer. Hence in the initial growth of the thermal layer, convection is effected by the velocity components near the wall and so the correct procedure is to replace the velocity components by their values near the wall. This procedure, carried out in this paper, shows that the correct series expansion for small τ is one in powers of $\tau^{\frac{3}{2}}$. The first term of this series, identical with that given by Cess, is due to diffusion, which is already in progress before the effects of convection become significant.

For large values of τ Cess uses a Laplace transform technique and concludes that the steady state is reached abruptly after a finite time at each station x.

Whilst the steady-state solution given is the appropriate one for small values of the Prandtl number it is difficult to believe that, in any process which involves diffusion, a steady state can be attained suddenly in this way. It is shown in § 4 below that in the final decay to a steady state the departure from the steady state is ultimately concentrated near the wall. The velocity components are therefore again replaced by their values near the wall. Although the final decay to the steady state cannot be determined by means of a linear perturbation of the separable type the steady-state solution is shown to be approached in an exponential manner.

The problem of final decay to the steady state is closely analogous to that involved when a semi-infinite flat plate is impulsively set in motion, parallel to itself, with constant velocity. Stewartson (1951) and subsequently Kelly (1962) have shown that the approach to steady state in that problem cannot be determined by means of a perturbation of the separable type. Stewartson shows that, as in the present paper, the steady state is approached exponentially and in the final decay the departure from it is concentrated near the wall.

2. Equations of motion

The momentum and continuity equations for the boundary-layer problem described in § 1 are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0. \tag{2}$$

Here the co-ordinates (x,y) are measured along and normal to the plate respectively and (u,v) represent the velocity components in these directions; the kinematic viscosity ν is assumed to be constant. These equations are independent of the time t since changes in density, associated with changes in the plate temperature, are assumed to be so small as not to affect the velocity boundary layer already established upon the plate.

The solution of (1) and (2) under the boundary conditions

$$\left.\begin{aligned} &u = v = 0, \quad y = 0, \quad x > 0;\\ &u \to U_\infty, \quad y \to \infty;\\ &u = U_\infty, \quad x = 0, \quad y > 0;\end{aligned}\right\} \tag{3}$$

is the well-known Blasius solution given by

$$u = U_\infty f'(\eta), \quad v = \tfrac{1}{2}(\nu U_\infty/x)^{\frac{1}{2}}\{\eta f'(\eta)-f(\eta)\}, \tag{4}$$

where the similarity variable η is defined as

$$\eta = y(U_\infty/\nu x)^{\frac{1}{2}}, \tag{5}$$

and f satisfies the equation

$$\left.\begin{aligned} &f''' + \tfrac{1}{2}ff'' = 0,\\ \text{with}\qquad &f(0)=f'(0) = 0, \quad f'(\infty) = 1.\end{aligned}\right\} \tag{6}$$

The energy equation satisfied by the temperature T is

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\kappa\frac{\partial^2 T}{\partial y^2}, \tag{7}$$

where κ, the thermal diffusivity, is assumed to be constant. The heating due to viscous dissipation is neglected in the formulation of (7) as the Mach number M_∞ of the flow is assumed to be small. The boundary conditions satisfied by T are

$$\left.\begin{aligned} T &= T_\infty, \quad t=0, \quad y>0;\\ T &= T_\infty, \quad x=0, \quad y>0;\\ T &= T_w, \quad y=0, \quad t>0, \quad x>0;\\ T &\to T_\infty, \quad y\to\infty. \end{aligned}\right\} \tag{8}$$

Here T_w and T_∞ are the constant plate and free-stream temperatures and for consistency we require $M_\infty^2 \ll (T_w - T_\infty)/T_\infty \ll 1$.

At time $t=0$, when the plate temperature is suddenly changed, the velocity boundary layer is already established on the plate and the solution of (1) and (2) is given by equations (4)–(6). To solve equation (7) for the temperature we introduce new dimensionless variables

$$\theta = \frac{T-T_\infty}{T_w-T_\infty}, \tag{9}$$

and

$$\tau = U_\infty t/x. \tag{10}$$

In terms of the new variables θ, τ and η (7) becomes

$$\frac{\partial^2\theta}{\partial\eta^2}+\tfrac{1}{2}\sigma f\frac{\partial\theta}{\partial\eta}+\sigma\left(\tau\frac{\partial f}{\partial\eta}-1\right)\frac{\partial\theta}{\partial\tau}=0, \tag{11}$$

where $\sigma=\nu/\kappa$ is the Prandtl number. Equation (11) is the appropriate form of the energy equation for studying the final decay to steady state. During the initial stages of growth of the thermal boundary layer, when the effects of thermal diffusion are important, we take a new independent variable

$$\zeta = \tfrac{1}{2}y/(\kappa t)^{\frac{1}{2}} = \tfrac{1}{2}\eta(\sigma/\tau)^{\frac{1}{2}}. \tag{12}$$

Written in terms of the variables (ζ, τ) the energy equation now becomes

$$\frac{\partial^2\theta}{\partial\zeta^2}+\left\{(\tau\sigma)^{\frac{1}{2}}\left(f-\zeta\frac{\partial f}{\partial\zeta}\right)+2\zeta\right\}\frac{\partial\theta}{\partial\zeta}+2\tau\left\{(\tau\sigma)^{\frac{1}{2}}\frac{\partial f}{\partial\zeta}-2\right\}\frac{\partial\theta}{\partial\tau}=0. \tag{13}$$

The boundary conditions for equations (11) and (13) are

$$\left.\begin{aligned} \theta &= 0, \quad \tau=0, \quad \eta>0, \quad \zeta=\infty;\\ \theta &= 1, \quad \tau>0, \quad \eta,\zeta=0;\\ \theta &\to 0, \quad \tau>0, \quad \eta,\zeta\to\infty. \end{aligned}\right\} \tag{14}$$

3. Solution of the energy equation for small τ

As anticipated in § 1 we assume that for sufficiently small values of τ the thermal boundary layer, growing within the velocity boundary layer, is very much thinner than the latter. The velocity function f may therefore be replaced by its series expansion near the wall, in equation (13), when developing the solution for small τ. Eventually, for sufficiently large τ, this assumption will not necessarily be true and in particular if $\sigma \ll 1$ the thermal boundary layer will ultimately become much thicker than the velocity boundary layer. It can be shown that for small η

$$f = \tfrac{1}{2}\alpha\eta^2 - \tfrac{1}{240}\alpha^2\eta^5 + O(\eta^8) \quad \text{where} \quad \alpha = 0{\cdot}332, \tag{15}$$

and so, retaining only the first two terms of the series for f in equation (13) we have

$$\frac{\partial^2\theta}{\partial\zeta^2} + 2\zeta\frac{\partial\theta}{\partial\zeta} - 4\tau\frac{\partial\theta}{\partial\tau} = \frac{2\alpha}{\sigma^{\frac{1}{2}}}\zeta^2\tau^{\frac{3}{2}}\frac{\partial\theta}{\partial\zeta} - \frac{8\alpha}{\sigma^{\frac{1}{2}}}\zeta\tau^{\frac{5}{2}}\frac{\partial\theta}{\partial\tau} - \frac{8\alpha^2}{15\sigma^2}\zeta^5\tau^3\frac{\partial\theta}{\partial\zeta} + \frac{4}{3}\frac{\alpha^2}{\sigma^2}\zeta^4\tau^4\frac{\partial\theta}{\partial\tau} + O(\tau^{\frac{9}{2}}). \tag{16}$$

Consider now the possibility of a solution of equation (16) in the form

$$\theta(\zeta,\tau) = \sum_p \tau^p\Theta_p(\zeta) \quad (p \geqslant 0). \tag{17}$$

From (16) and (17) we see that if $p > 0$ and $\tfrac{2}{3}p$ is not a positive integer then $\Theta_p(\zeta)$ satisfies

$$\left.\begin{aligned} \Theta_p'' + 2\zeta\Theta_p' - 4p\Theta_p &= 0, \\ \Theta_p(0) = \Theta_p(\infty) &= 0, \end{aligned}\right\} \tag{18}$$

and hence Θ_p is related to the parabolic cylinder function $D_n(z)$. A solution of (18) which tends to zero as $\zeta \to \infty$ is

$$\Theta_p = \exp\left(-\tfrac{1}{2}\zeta^2\right) D_{-2p-1}(\zeta\sqrt{2}), \tag{19}$$

but this does not satisfy the boundary condition at the origin. We conclude that a series solution of equation (16) in powers of τ must be of the form

$$\theta(\zeta,\tau) = \sum_{r=0}^{\infty} \tau^{\frac{3}{2}r}\phi_r(\zeta), \tag{20}$$

where r is an integer and the functions $\phi_r(\zeta)$ satisfy inhomogeneous ordinary differential equations, the homogeneous part of these equations being of the form (18). The series (20) may be compared with the series solution obtained by Cess in powers of $\tau^{\frac{1}{2}}$. The equations satisfied by the functions ϕ_0, ϕ_1 and ϕ_2 are

$$\phi_0'' + 2\zeta\phi_0' = 0, \tag{21}$$

$$\phi_1'' + 2\zeta\phi_1' - 6\phi_1 = \frac{2\alpha}{\sigma^{\frac{1}{2}}}\zeta^2\phi_0', \tag{22}$$

$$\phi_2'' + 2\zeta\phi_2' - 12\phi_2 = \frac{2\alpha}{\sigma^{\frac{1}{2}}}(\zeta^2\phi_1' - 6\zeta\phi_1) - \frac{8\alpha^2}{15\sigma^2}\zeta^5\phi_0', \tag{23}$$

together with the boundary conditions

$$\left.\begin{aligned} \phi_0(0) = 1; \qquad \phi_r(0) = 0 \quad (r > 0); \\ \phi_r(\infty) = 0 \quad (\text{all } r). \end{aligned}\right\} \tag{24}$$

Solutions of these equations, subject to (24), are

$$\phi_0 = \operatorname{erfc}\zeta, \tag{25}$$

$$\phi_1 = \frac{\alpha}{12\sqrt{(\pi\sigma)}}\left\{3\zeta^2 e^{-\zeta^2} + \frac{\sqrt{\pi}}{2}(3\zeta+2\zeta^3)\operatorname{erfc}\zeta\right\}, \tag{26}$$

$$\phi_2 = \frac{\alpha^2}{6\sigma\sqrt{\pi}}\left\{\left[\left(\frac{121\sqrt{\pi}}{160}-\frac{17}{20}\right)\zeta + \left(\frac{77\sqrt{\pi}}{120}+\frac{29}{60}\right)\zeta^3 + \left(\frac{11\sqrt{\pi}}{120}+\frac{1}{3}\right)\zeta^5\right]e^{-\zeta^2} - \frac{\sqrt{\pi}}{4}\left(\frac{5\zeta^4}{2}+\frac{11\zeta^6}{15}\right)\operatorname{erfc}\zeta\right\} - \frac{2\alpha^2}{15\sigma^2\sqrt{\pi}}\left(\frac{\zeta}{8}+\frac{\zeta^3}{3}+\frac{\zeta^5}{3}\right)e^{-\zeta^2}. \tag{27}$$

Further terms of this series may be obtained by retaining more terms of the series for f in equation (13); at each stage the details become more tedious.

The heat-transfer rate across the wall per unit area, Q, is given by

$$\frac{Qx}{\lambda(T_w - T_\infty)R^{\frac{1}{2}}} = -\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0}, \tag{28}$$

where $R = U_\infty x/\nu$ and λ is the thermal conductivity. Hence, from (12), (20), (25), (26) and (27), we have

$$\frac{Qx}{\lambda(T_w - T_\infty)\sqrt{R}} = 0{\cdot}564\sigma^{\frac{1}{2}}\tau^{-\frac{1}{2}} - 0{\cdot}0208\tau - 0{\cdot}00052\sigma^{-\frac{3}{2}}(4{\cdot}9\sigma - 1)\tau^{\frac{5}{2}} + O(\tau^4). \tag{29}$$

The first term of (29) is identical with the first term of the series given by Cess for the heat transfer and represents the heat transfer due to pure diffusion.

4. Solution of the energy equation for large τ

For large values of τ we consider the energy equation in the form (11). The steady-state solution $\theta_0(\eta)$ is obtained by neglecting the time-dependent term in (11); this gives as the equation for θ_0

$$\theta_0'' + \tfrac{1}{2}\sigma f \theta_0' = 0, \tag{30}$$

with

$$\theta_0(0) = 1, \quad \theta_0(\infty) = 0. \tag{31}$$

The solution of (30) under the boundary conditions (31) is well known and may be written in the form

$$\theta_0 = F(\eta)/F(0), \tag{32}$$

where

$$[f''(0)]^\sigma\, F(\eta) = \int_\eta^\infty [f''(v)]^\sigma\, dv. \tag{33}$$

Cess concludes from his analysis that the steady-state solution given by (32) is reached abruptly after a finite time at each station x. As stated in § 1 it does not seem reasonable to expect that in any process which involves diffusion a steady state will be reached suddenly in this way but rather for each x it will be approached gradually as $t \to \infty$. Consequently we now seek a perturbation of the steady solution (32) to investigate how this steady state is realized. Write

$$\theta(\eta,\tau) = \theta_0(\eta) + \theta_1(\eta,\tau), \tag{34}$$

where θ_1 satisfies

$$\frac{\partial^2\theta_1}{\partial\eta^2} + \tfrac{1}{2}\sigma f\frac{\partial\theta_1}{\partial\eta} + \sigma\left(\tau\frac{\partial f}{\partial\eta} - 1\right)\frac{\partial\theta_1}{\partial\tau} = 0, \tag{35}$$

with
$$\left.\begin{aligned}\theta_1(0,\tau) &= 0,\\ \theta_1(\infty,\tau) &= 0.\end{aligned}\right\} \tag{36}$$

If we assume that this perturbation is separable so that
$$\theta_1(\eta,\tau) = \chi(\tau)\,\Phi(\eta), \tag{37}$$
then, from (35) and (37),
$$\Phi'' + \tfrac{1}{2}\sigma f\,\Phi' + \sigma(\tau f' - 1)\,\Phi\dot{\chi}/\chi = 0, \tag{38}$$
where the primes denote differentiation with respect to η and the dots differentiation with respect to τ. We see from equation (38) that $\theta_1(\eta,\tau)$ can only be written in the form (37) provided that one of the two terms in the brackets in the last term of (38) may be neglected. However, since $f'(0) = 0$, no matter how large τ becomes there will always be a region, near the wall, in which $\tau f' = O(1)$; consequently we are not justified in neglecting either of these terms and we conclude that $\theta_1(\eta,\tau)$ may not be written in the form (37).

In order to investigate the asymptotic form of $\theta_1(\eta,\tau)$ for large τ note that the boundary conditions (36) imply that θ_1 has a point of inflexion and that, from (35), this inflexion is in the neighbourhood of $\tau f' = 1$ or, for large τ, $\tau\eta = O(1)$. Thus, in the final decay to steady state, departures from the steady state are concentrated near the wall and so, in equation (35), f and f' may again be replaced by the first terms of their series expansions near the wall given by (15). In the analogous problem of the flat plate set in motion impulsively a similar conclusion was arrived at by Stewartson. Watson has recently extended Stewartson's work and the author is indebted to him for an opportunity to examine this unpublished work. Defining a new variable $\xi = \tau\eta$ and writing equation (35) in terms of (ξ,τ) we have, retaining only the first two terms in the series for f and f', as the equation for θ_1
$$\tau^2\frac{\partial^2\theta_1}{\partial\xi^2} + \tfrac{1}{2}\sigma\left(\frac{\alpha\xi^2}{2\tau} - \frac{\alpha^2\xi^5}{240\tau^4}\right)\frac{\partial\theta_1}{\partial\xi} + \sigma\left(\alpha\xi - \frac{\alpha^2\xi^4}{48\tau^3} - 1\right)\left(\frac{\partial\theta_1}{\partial\tau} + \frac{\xi}{\tau}\frac{\partial\theta_1}{\partial\xi}\right) = O(\tau^{-7}\theta_1), \tag{39}$$
with
$$\theta_1(0,\tau) = \theta_1(\infty,\tau) = 0. \tag{40}$$

We now assume a solution of equation (39) in the form
$$\theta_1(\xi,\tau) = AT(\tau)\,Z(\xi,\tau), \tag{41}$$
where A is a constant and, as $\tau \to \infty$,
$$Z(\xi,\tau) \sim \sum_{n=0}^{\infty} Z_n(\xi)\,\tau^{-n}. \tag{42}$$

Substituting in equation (39) for θ_1 from (41) and (42) we see that $T(\tau)$ must be given from an equation of the form
$$T'/T = -\beta\tau^2 + a\tau + b + c\tau^{-1} + \ldots, \tag{43}$$
where β, a, b and c are constants to be determined, giving
$$T = \exp\left(-\tfrac{1}{3}\beta\tau^3 + \tfrac{1}{2}a\tau^2 + b\tau + c\log\tau\right). \tag{44}$$

No further terms in (43) and (44) need be considered as smaller terms may be absorbed into $Z(\xi,\tau)$. The functions $Z_n(\xi)$ must satisfy, from (40),
$$Z_n(0) = Z_n(\infty) = 0, \tag{45}$$

and, in terms of a new variable

$$s = (\beta\sigma/\alpha^2)^{\frac{1}{3}}(\alpha\xi - 1), \tag{46}$$

the functions Z_0, Z_1 and Z_2 satisfy

$$Z_0'' - sZ_0 = 0, \tag{47}$$

$$Z_1'' - sZ_1 = -saZ_0/\beta, \tag{48}$$

$$Z_2'' - sZ_2 = -s(aZ_1 + bZ_0)/\beta, \tag{49}$$

where the primes now denote differentiation with respect to s. The solution of equation (47) which satisfies the second of the boundary conditions (45) is

$$Z_0 = \mathrm{Ai}\,(s), \tag{50}$$

where $\mathrm{Ai}\,(s)$ denotes the Airy function. The determination of β is an eigenvalue problem which is solved by satisfying the boundary condition at $\xi = 0$. For $s > 0$, $\mathrm{Ai}\,(s)$ has no zeros for finite s but for $s < 0$, $\mathrm{Ai}\,(s)$ has an infinite number of zeros at $s = -s_n$ with

$$\mathrm{Ai}\,(-s_n) = 0, \quad \mathrm{Ai}'\,(-s_n) \neq 0. \tag{51}$$

For the largest contribution to $T(\tau)$ in equation (41) we take, to satisfy the first of the boundary conditions (45), the smallest value, s_1, for which equation (51) holds. Thus

$$s_1 = 2{\cdot}338, \tag{52}$$

and the corresponding value of β is

$$\beta_1 = s_1^3\alpha^2/\sigma = 1{\cdot}409\sigma^{-1}. \tag{53}$$

With Z_0 given by equation (50) the solution of (48) which satisfies the second of the conditions (45) is

$$Z_1 = -(a/3\beta_1)\,s\,\mathrm{Ai}'\,(s) + C_1\,\mathrm{Ai}\,(s), \tag{54}$$

where C_1 is a constant. The first of the conditions (45) together with equation (51) shows that a must be zero. A solution of (49) is similarly given as

$$Z_2 = -(b/3\beta_1)\,s\,\mathrm{Ai}'\,(s) + C_2\,\mathrm{Ai}\,(s), \tag{55}$$

where C_2 is a constant, and the same considerations show that $b = 0$.

Consider now the equations for Z_3 and Z_4; these are

$$Z_3'' - sZ_3 = -\frac{\sigma}{4\alpha^2 s_1}\left(1+\frac{s}{s_1}\right)^2 Z_0' - \frac{\sigma c}{\alpha^2 s_1^2}\frac{s}{s_1}Z_0 - \frac{s_1}{48\alpha^2}\left(1+\frac{s}{s_1}\right)^4 Z_0 - \frac{\sigma}{\alpha^2 s_1}\left(1+\frac{s}{s_1}\right)\frac{s}{s_1}Z_0', \tag{56}$$

since $\sigma\beta_1 = \alpha^2 s_1^3$, and

$$Z_4'' - sZ_4 = C_1(Z_3'' - sZ_3) + C_1 sZ_0/\beta_1, \tag{57}$$

since $Z_1 = C_1 Z_0$. Consider first equation (57) for Z_4. The solution of this equation satisfying the second of (45), assuming that Z_3 satisfies both the conditions (45), is given by

$$Z_4 = C_1 Z_3 + (C_1/3\beta_1)\,s\,\mathrm{Ai}'\,(s) + C_4\,\mathrm{Ai}\,(s), \tag{58}$$

where C_4 is a further constant. To satisfy the first of the boundary conditions (45) we must have, from (51), $C_1 = 0$. An investigation of the equation for Z_5 shows similarly that C_2 is zero. Consider finally the equation (56) for Z_3. The solution of this equation enables us to determine the constant c occurring in equations (43) and (44). Thus

$$Z_3 = \bar{Z}_3 + C_3\,\mathrm{Ai}\,(s), \tag{59}$$

where C_3 is a constant and $\bar{Z}_3(s)$ is a particular solution of (56) given by

$$\begin{aligned} -4\alpha^2\bar{Z}_3 = \frac{\mathrm{Ai}\,(s)}{2}\frac{s}{s_1}\left\{\left(\sigma-\frac{1}{5}\right)+\left(3\sigma-\frac{1}{7}\right)\frac{s}{s_1}+\frac{1}{3}\left(5\sigma-\frac{1}{9}\right)\frac{s^2}{s_1^2}\right\} \\ +\frac{\mathrm{Ai}'\,(s)}{s_1^2}\left\{\frac{1}{7}-3\sigma+\frac{1}{3}\left[\frac{1}{9}+(4c-5)\,\sigma\right]\frac{s}{s_1}\right. \\ \left.+s_1^3\left(\frac{1}{12}+\frac{1}{9}\frac{s}{s_1}+\frac{1}{10}\frac{s^2}{s_1^2}+\frac{1}{21}\frac{s^3}{s_1^3}+\frac{1}{108}\frac{s^4}{s_1^4}\right)\right\}. \end{aligned} \tag{60}$$

The solution (59) satisfies the second of the boundary conditions (45) and the first of these conditions requires

$$\bar{Z}_3(-s_1) = 0. \tag{61}$$

Equation (61) then determines c_1, the value of c appropriate to s_1, as

$$\sigma(1+c_1) = \frac{8s_1^3}{315}+\frac{5}{63} = 0{\cdot}404. \tag{62}$$

The constant C_3 occurring in equation (59) is determined when the solution for Z_6 is investigated and is non-zero. Thus, as $\tau \to \infty$, the dominant term in θ_1 is given by

$$\theta_1 = A_1\tau^{c_1}\exp\left(-\tfrac{1}{3}\beta_1\tau^3\right)\{\mathrm{Ai}\,(s)+O(\tau^{-3})\}, \tag{63}$$

where β_1 and c_1 are given from equations (53) and (62) respectively. There is such a solution for each s_n satisfying (51) but the solution given in (63) is the most important. The constant A_1 will depend, in some way, upon the initial growth of the thermal boundary layer. The heat transfer parameter, defined in equation (28), is given by

$$\frac{Qx}{\lambda(T_w-T_\infty)\,R^{\frac{1}{2}}} = [F(0)]^{-1} - 0{\cdot}544A_1\tau^{1+c_1}\exp\left(-\tfrac{1}{3}\beta_1\tau^3\right)\{1+O(\tau^{-3})\}. \tag{64}$$

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